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Optimal Feedback Control of Infinite Dimensional Linear Systems with Applications to Hereditary Problems

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DEDICATED TO THE MEMORY OF RICHARD BELLMAN

1. INTRODUCTION

This paper develops an operator theoretic approach to optimal feedback control of linear systems with quadratic cost. Originally intended as an applications oriented sequel to [1], recent developments [2] enable us to streamline the arguments and results in [1] and also to circumvent many of the technical aspects involved in applying those results. However, the focal point of the paper is the application of the general theory to dynamical systems governed by retarded functional differential equations. In this example we also allow virtually unconstrained control delays. The results for this case were originally obtained (by forerunners of the present

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approach) in the dissertations of Foster [3] and Milman [4]. In the present paper, as in [3, 4], the "input-output" representation of the system is emphasized in the feedback formulation. On the other hand, a "state space" formulation of the hereditary control problem with control delays can also be considered. Research in this direction has appeared in [12, 19, 20, 21].

Our approach to the optimal feedback synthesis problem for the example systems above is to initially treat an abstract control problem, which is introduced at a level of generality that precludes what one would call a state feedback representation of the control law. The basic elements of this problem are an input space (a Hilbert space), an output space (a Banach space), a forcing term, and a pair of bounded causal operators. The cost functional is formulated in terms of the inner product on the input space and a nonnegative definite bilinear functional on the output space. As defined the problem possesses enough structure to determine the optimal control solution, although there is not quite enough structure to construct feedback solutions. However, the causality of the maps connecting the input and output permits us to develop a "principle of optimality," from which a control law that is intermediate between open and closed loop is derived. For lack of a better term we call this quasifeedback. Heuristically this can be described as a feedback on the space of trajectories of the system (which does have meaning even though the state of the system is not explicitly defined). In applications to specific systems the feedback control law is then derived from the quasifeedback representation.

The approach above brings new emphasis to the role of the "input-output" map in optimal feedback representations and also leads to a clarification of some of the connections between the principle of optimality, open and closed loop optimal controllers, Wiener–Hopf equations, and the Volterra factorization of operators (see also [5, 6]). Although the methods of the paper are applicable to other classes of systems, the hereditary control problem is particularly illustrative of these features. This point will be discussed in greater detail in Section 3.

Before leaving this introduction, one last historical footnote is in order. Two of the main "principles" used in the above approach are:

(1) the "principle of optimality" (cf., Theorem 2.6) and

(2) the "principle of Volterra factorization" (cf., Theorem 1.3).

It is well known that the original "principle of optimality" was discovered by Richard Bellman in 1953, and was the cornerstone of the method of dynamic programming [22].

What is not as well known is that Richard Bellman also discovered (independently) the first application of the "principle of Volterra factorization" in his paper on "A Partial Differential Equation for the Fredholm Resolvent," published in 1957 [23]. (Results similar to [23] appeared independently and approximately at the same time in the fields of scattering theory [24], general integral equations [25], and random processes [26].)

1. BACKGROUND

We will make liberal use of the results from [1, 2] throughout this paper. In this section we will briefly review some of the more basic results from these references.

For any Hilbert spaces H and K, let $B(K, H)$ denote the space of all bounded linear maps from K into H and $B(H) = B(H, H)$.

Let $[t_0, t_f]$ be a closed bounded interval on the real line and let Σ denote the class of Borel subsets of $[t_0, t_f]$. Let $E\colon \Sigma \to B(H)$ denote an "absolutely continuous" resolution of the identity, i.e., $|E(\omega)\, x|^2 = 0$ if $\lambda(\omega) = 0$, where λ is Lebesgue measure ($|\cdot|$ will always denote a norm). All resolutions of the identity defined in the sequel will be absolutely continuous. Given a finite positive Borel measure μ, and a map $T \in B(K, H)$, we shall say that T is dominated by μ (written $T < \mu$) if there exists $\gamma \geqslant 0$ such that $|E(\omega)\, T|^2 \leqslant \gamma\, \mu(\omega)$ for all $\omega \in \Sigma$. The space of all such maps is written L^μ and can be topologized so that it is a Banach space via the norm

$$|T|_\mu = \inf\{\gamma\colon |E(\omega)\, T| \leqslant \gamma \sqrt{\mu(\omega)} \text{ for all } \omega \in \Sigma\}. \tag{1.1}$$

When H and K coincide, L^μ forms a right ideal in $B(H)$.

Define $\mathbf{K}^\mu = L_2([t_0, t_f], K; \mu)$ as the space of μ-square integrable functions on $[t_0, t_f]$ with values in K. Let E' denote the truncation resolution on $\mathbf{K}^\mu$, i.e., $(E'(\omega)\, x)(t) = \chi(\omega)(t)\, x(t)$ where χ is the characteristic function. Then the space of "memoryless" maps $\mathbf{M}(\mathbf{K}^\mu, H)$,

$$\mathbf{M}(\mathbf{K}^\mu, H) = \{T \in B(\mathbf{K}^\mu, H)\colon E(\omega)\, T = TE'(\omega) \text{ for all } \omega \in \Sigma\} \tag{1.2}$$

is isometrically isomorphic to L^μ (see [2]). Furthermore this isomorphism is explicitly described by the mapping $\mathbf{F}_\mu\colon L^\mu \to \mathbf{M}(\mathbf{K}^\mu, H)$ where for each $T \in L^\mu$ and $x \in \mathbf{K}^\mu$ simple, say $x(t) = \sum_{i=1}^n \chi(\omega_i)(t)\, x_i$ with $x_i \in K$,

$$\mathbf{F}_\mu(T)\, x = \sum_{i=1}^{n} E(\omega_i)\, Tx_i. \tag{1.3}$$

$\mathbf{F}_\mu(T)$ is then extended to the entire space by continuity.

The evaluation of this mapping plays an important role in the sequel. We now present the general situation in which it will arise.

Let H_1 and H_2 denote two real separable Hilbert spaces. Let $H = L_2([t_0, t_f], H_1; \lambda)$ with the truncation resolution, $(E(\omega)\,x)(t) = \chi(\omega)(t)\,x(t)$ and define $K = L_2([t_0, t_f], H_2; \nu)$ where ν is an arbitrary finite positive Borel measure. Define $T \in B(K, H)$ by

$$Tx\colon t \to \int_{[t_0, t_f]} T(t, s)\,x(s)\,d\nu(s) \tag{1.4}$$

where $T(t, s)$ is strongly measurable and

$$\int_{t_0}^{t_f} \int_{t_0}^{t_f} |T(t, s)|^2 d\nu(s)\,dt < \infty. \tag{1.5}$$

PROPOSITION 1.1. *Define the measure μ on Σ by*

$$\mu(\omega) = \int_\omega \left(\int_{t_0}^{t_f} |T(t, s)|^2\,d\nu(s) \right) dt.$$

Let $\mathbf{K}^\mu = L_2([t_0, t_f], K; \mu)$ with the truncation resolution. Then for $x \in \mathbf{K}^\mu$,

$$[\mathbf{F}_\mu(T)\,x](t) = \int_{t_0}^{t_f} T(t, s)\,x(t)(s)\,d\nu(s), \tag{1.6}$$

with the convention that the integral is zero if $|x(t)|_K = \infty$ and $\int |T(t, s)|^2\,d\nu(s) = 0$.

Proof. Let x_n be a sequence of simple functions in $\mathbf{K}^\mu$ such that $x_n \to x$ (in $\mathbf{K}^\mu$) and $x_n(t) \to x(t)$ μ-a.e. Also we choose the x_n so that $x_n(t) = 0$ on the μ-negligible set $Z = \{t\colon |x(t)|_K = \infty\}$ and $x_n \to x$ pointwise on Z^c. Let

$$g(t) = \int |T(t, s)|^2\,d\nu(s)$$

and define $Z_0 = \{t \in Z\colon g(t) = 0\}$. By definition

$$\begin{aligned} \mathbf{F}_\mu(T)\,x_n &= \sum E(\omega_{n_i})\,Tx_{n_i} \\ &= \sum \chi(\omega_{n_i})(t) \int T(t, s)\,x_{n_i}(s)\,d\nu(s) \\ &= \int T(t, s)\,x_n(t)(s)\,d\nu(s), \end{aligned}$$

where $x_n(t)=\Sigma\, \chi(\omega_{n_i})(t)\, x_{n_i}$ with $x_{n_i}\in K$. Now define $\psi(t)$ by the right-hand side of (1.6). It follows that

$$0=\psi(t)=[\mathbf{F}_\mu(T)\, x_n](t), \qquad t\in Z_0.$$

Now for $t\in Z^c$

$$\begin{aligned} |\psi(t)-(\mathbf{F}_\mu(T)\, x_n)(t)|^2 &= \left|\int T(t,s)\{x(t)(s)-x_n(t)(s)\}\, dv(s)\right|^2 \\ &\leqslant \int |T(t,s)|^2\, dv(s) \int |x(t)(s)-x_n(t)(s)|^2\, dv(s) \\ &\leqslant g(t)|x(t)-x_n(t)|_K^2. \end{aligned} \tag{1.7}$$

Thus

$$\psi(t)=\lim_n\, [\mathbf{F}_\mu(T)\, x_n](t), \qquad t\in Z^c\cap\{t\colon g(t)<\infty\}.$$

Next note that $\lambda\{Z-Z_0\}=0$ since $g(t)>0$ on $Z-Z_0$ and

$$0=\mu(Z)=\int_{Z_0} g(t)\, dt+\int_{Z-Z_0} g(t)\, dt.$$

Then since $\lambda\{Z\cup(Z^c\cap\{t\colon g(t)<\infty\})^c\}=0$, we obtain $\psi(t)=\lim_n\, [\mathbf{F}_\mu(T)\, x_n](t)$ λ-a.e. By continuity of $\mathbf{F}_\mu(T)$, $\lim_n \mathbf{F}_\mu(T)\, x_n=\mathbf{F}_\mu(T)\, x$, and the proposition is proved. ▌

We note that since μ is finite, $C([t_0,t_f],K)\subsetneqq \mathbf{K}^\mu$, so that the evaluation above is valid for continuous functions.

Two important classes of operators of concern in the sequel are the classes of causal and anticausal operators. Thus let H and K denote Hilbert spaces with resolutions of the identity E_1 and E_2, respectively. Define the projections $P_i^t=E_i([0,t])$, $i=1,2$. A map $T\in B(K,H)$ is causal if $P_1^t\, TP_2^t=P_1^t\, T$ for all $t\in[t_0,t_f]$ and anticausal if $(I-P_1^t)\, T(I-P_2^t)=(I-P_1^t)\, T$ for all $t\in[t_0,t_f]$. In [2] is is shown that there exist continuous projections $p^\pm$ on L^μ such that each $T\in L^\mu$ has the unique decomposition

$$T=p^+(T)+p^-(T)$$

with $p^+(T)$ causal and $p^-(T)$ anitcausal. It is further shown in [2] that if μ is absolutely continuous with respect to λ, written $\mu\ll\lambda$, then $p^\pm(T)$ are quasinilpotents. For brevity we shall write $T_\pm$ for $p^\pm(T)$.

The principal tool to be used in the derivation of the control laws is the Volterra factorization of operators. Specifically the Volterra (or special) factorization problem is: Given $T \in B(H)$, find $X_+, X_- \in B(H)$, respectively causal and anticausal such that

$$(I+T)=(I+X_-)(I+X_+).$$

The main theorem on factorization from [2] is

THEOREM 1.3. *Let $T, T^* \in L^\mu$, with $\mu \ll \lambda$. Then there exist unique $X_\pm \in L^\mu$ with $(X_\pm)^* \in L^\mu$ such that*

$$(I+T)=(I+X_-)(I+X_+) \tag{1.8}$$

if and only if $(I+P_tTP_t)^{-1} \in B(H)$ for each $t \in [t_0, t_f]$.

From the uniqueness of the factorization, it follows that if T is self-adjoint then $X_- = (X_+)^*$.

2. Control Problem Statement and Quasifeedback Representation

In this section we introduce an abstract control problem and synthesize its solution in the form that will serve as the departure point for the subsequent example.

Let $[t_0, t_f]$ be a closed bounded interval on the real line. Let X denote a real Banach space and let U denote a real Hilbert space. We shall assume given a family of bounded projections on X, $\{P_X^t\}$, $t \in [t_0, t_f]$, such that $P_X^{t_2} P_X^{t_1} = P_X^{t_1}$ for $t_1 \leq t_2$, and further that $t \to P_X^t$ is strongly continuous. Similarly we shall assume given a family of strongly continuous orthoprojectors on U, $\{P_U^t\}$, $t \in [t_0, t_f]$, such that $P_U^{t_2} P_U^{t_1} = P_U^{t_1}$ and $P_U^{t_0} = 0$, $P_U^{t_f} = I$. The complementary projections P_t on these spaces are defined as $P_t = I - P^t$. (The subscripts on these projections will be suppressed when no ambiguities arise.) We note that the family of orthoprojectors on U generates a unique resolution of the identity E on the Borel subsets of $[t_0, t_f]$, such that $E([t_1, t_2]) = P^{t_2} - P^{t_1}$ for all t_1, t_2.

The class of systems we consider in this section is of the form

$$x = f + Lx + Tu \tag{2.1}$$

with $f, x \in X$, $u \in U$, $L \in B(X)$, and $T \in B(U, X)$. In the above, f represents a prescribed forcing term, u represents the control input, and x is the output of the system. The maps L and T are assumed causal, i.e., $P^t L P^t = P^t L$ and $P^t T P^t = P^t T$ for all $t \in [t_0, t_f]$. We shall also assume that $P^{t_0} L = 0$. The cost functional is defined as

$$J(u, x) = \tfrac{1}{2}\langle x, x\rangle_X + \tfrac{1}{2}\langle u, u\rangle \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ is the inner product on U and $\langle \cdot, \cdot \rangle_X$ is a bounded symmetric nonnegative definite bilinear form on X.

The objective in this section is to determine a "quasifeedback" form of the control $u \in U$ that minimizes (2.2) subject to the constraint (2.1). This form of the solution will be achieved by considerations involving the following family of embedded problems: For $t \in [t_0, t_f]$,

$$\min_{x,u} J(u, x) = \tfrac{1}{2}\langle x, x\rangle_X + \tfrac{1}{2}\langle u, u\rangle \tag{2.3}$$

subject to the constraint

$$x = f_t + P_t Lx + TP_t u. \tag{2.4}$$

The term $f_t \in X$ above has not yet been prescribed, but note that (2.4) is equivalent to (2.1) when $t = t_0$ and $f_{t_0} = f$.

Before we can state the hypotheses under which the control law takes the desired form, the following proposition is needed.

PROPOSITION 2.1. *Given a map $S \in B(U, X)$, there exists a unique map $S^\# \in B(X, U)$ such that for all $u \in U$, $x \in X$*

$$\langle Su, x\rangle_X = \langle u, S^\# x\rangle. \tag{2.5}$$

Proof. Let X^* denote the dual of X and define the map $j\colon X \to X^*$ by $j(x) = \langle \cdot, x\rangle_X$. Since j is bounded, so is S^*j, where S^* is the Banach space adjoint of S (identifying U with its dual). Defining $S^\# = S^*j$, we see that

$$\langle u, S^\# x\rangle = \langle u, S^*(jx)\rangle = (jx)(Su) = \langle Su, x\rangle_X.$$

Uniqueness follows since (2.5) holds for all u and x. ∎

Note that the proposition implies that $S^\# S$ is self-adjoint and nonnegative.

HYPOTHESIS 2.2. *$G_t \equiv (I - P_t L)^{-1}$ exists and is causal for each $t \in [t_0, t_f]$. Furthermore the map $t \to G_t$ is continuous in $B(X)$. (Henceforth we shall write G for G_{t_0}.)*

HYPOTHESIS 2.3. *There exists a Hilbert space H and maps $\pi \in B(X, H)$, $A \in B(H, U)$ such that $(GT)^\# = A\pi$. Furthermore there exists a measure μ, absolutely continuous with respect to Lebesgue meausure, such that $A < \mu$ and $[(GT)^\# GT] < \mu$.*

A few words about the generality of these assumptions are appropriate. The first hypothesis is satisfied if L is Volterra in the sense of Neustadt

[15]. Note also that the case $L = 0$ does not lead to a degenerate control problem. The second hypothesis can be related to a condition on T which we give below.

PROPOSITION 2.4. *Suppose there exists a measure $\mu \ll \lambda$ and constant $\gamma \geq 0$ such that $\sup_{|u|=1} |T\,E(\omega)\,u|^2 < \gamma\,\mu(\omega)$ for all Borel subsets of $[t_0, t_f]$. Then there exists a Hilbert space H and maps $\pi \in B(X, H)$, $A \in B(H, U)$ such that $(GT)^\# = A\pi$ with $A < \mu$.*

Remarks. (i) Since $(GT)^\# = A\pi$ and L^μ is a right ideal, it follows that $(GT)^\#\, GT < \mu$.

(ii) Examples of operators satisfying the hypothesis of Proposition 2.4 are given in [2].

Proof. Let $\rho(\cdot) = \sqrt{\langle \cdot, \cdot \rangle_X}$. Since $\rho(\cdot)$ is a seminorm, it follows that the subspace $K = \{x: \langle x, x \rangle_X = 0\}$ is closed in X. Let $\pi: X \to X/K$ denote the canonical quotient map and define the bilinear form $\langle \cdot, \cdot \rangle_H$ on X/K by $\langle \pi x, \pi u \rangle_H = \langle x, y \rangle_X$. Since $\langle \pi x, \pi x \rangle_H = 0$ implies $\pi x = 0$, we have that $\langle \cdot, \cdot \rangle_H$ is an inner-product on X/K. Let H denote the completion of X/K with respect to $\langle \cdot, \cdot \rangle_H$. It follows from the definition of H that $\pi \in B(X, H)$. Now note that

$$\langle (\pi GT)^* \pi x, u \rangle_U = \langle \pi x, \pi GTu \rangle_H = \langle x, GTu \rangle_X.$$

We then have from Proposition 2.1 that $(GT)^\# = (\pi GT)^* \pi$. Now define $A = (\pi GT)^*$ and let ω denote a Borel subset of $[t_0, t_f]$. Then $|E(\omega)\, A|^2 = |A^* E(\omega)|^2 \leq |\pi|^2 |G|^2 |TE(\omega)|^2 \leq \gamma |\pi|^2 |G|^2 \mu(\omega)$, and $A < \mu$. From the definition of A it is clear that $A\pi = (GT)^\#$. ∎

From [1] we have the following result concerning the form of the open loop solution to the embedded optimization problem.

THEOREM 2.5. *The optimization problem* (2.3)–(2.4) *has the unique solution $\hat{u}_t = M_t f_t$ where*

$$M_t = -(I + P_t (GT)^\# (GT)\, P_t)^{-1} P_t (GT)^\# G_t \tag{2.6}$$

Proof. The proof of this result is essentially contained in [1]. We only need to note here that Proposition 2.1 implies that $P_t (GT)^\# GTP_t$ is nonnegative definite so that the inverse in Eq. (2.6) exists. ∎

In order to relate the solutions of the embedded problems to the original problem (2.1)–(2.2), it is necessary to choose the forcing term f_t correctly. This is provided by the principle of optimality given below.

THEOREM 2.6. *Let $\hat{u}$ denote the solution to* (2.1)–(2.2) *and let $\hat{x}$ denote the corresponding trajectory. Then for each $t \in [t_0, t_f]$,*

$$P_t \hat{u} = M_t f_t \tag{2.7}$$

where

$$f_t = P_t[TP^t \hat{u} + f] + P^t \hat{x} \tag{2.8}$$

Proof. [1] Now let $t_0 < t_1 < \cdots < t_n = t_f$ be an arbitrary partition of $[t_0, t_f]$ and let $\omega_i = [t_i, t_{i+1}]$. Then from Theorem 2.6 we obtain

$$\hat{u} = \sum_{i=0}^{n-1} E(\omega_i) M_{t_i} f_{t_i} \tag{2.9}$$

where $E(\omega_i) = P^{t_{i+1}} - P^{t_i}$, and M_{t_i} and f_{t_i} are defined by (2.6) and (2.8), respectively. In [1] a quasifeedback form of the controller was obtained by taking limits in (2.9) via a vector valued measure generated by the operators $E(\omega_i) M_{t_i}$. In the present approach we are able to circumvent several of the hypotheses in that paper and obtain the desired controller form in a more direct fashion. This is contained in the following basic result.

THEOREM 2.7. *Assume Hypotheses* (2.2) *and* (2.3). *Then $\hat{u}$ has the quasifeedback form*

$$\hat{u} = -\mathbf{F}_\mu((I + V_-)^{-1} A) z(\cdot)$$

where $z(\cdot) \in C([t_0, t_f], H)$, $z(t) = \pi G_t f_t$, and $V_- \in L^\mu$ is the anticausal Volterra factor of $(I + (GT)^\#(GT))$.

Proof. First note that $z(\cdot)$ as defined is continuous since $t \to G_t$ and $t \to f_t$ are continuous (using the strong continuity of the projections P_U^t and P_X^t). From Theorem 1.3, it follows that $(I + (GT)^\#(GT))$ has the unique factorization

$$(I + (GT)^\#(GT)) = (I + V_-)(I + V_+),$$

where $V_\pm \in L^\mu$, $(V_\pm)^* \in L^\mu$. Thus $(I + V_-) A \in L^\mu$. Now let $\varepsilon > 0$. Using the definition of $\mathbf{F}_\mu(\cdot)$, the continuity of $z(\cdot)$ and the absolute continuity of μ, we can find a partition $t_0 < t_1 < \cdots < t_n = t_f$ such that

$$|\mathbf{F}_\mu((I + V_-)^{-1} A) z(\cdot) - \sum_{i=0}^{n-1} E(\omega_i)(I + V_-)^{-1}(GT)^\# G_{t_i} f_{t_i}| < \varepsilon \tag{2.10}$$

and $\max_i \mu(\omega_i) < \varepsilon^2$ where $\omega_i = [t_i, t_{i+1}]$. Now since

$$(I + P_t(GT)^{\#}(GT)\, P_t) = (I + P_t V_- P_t)(I + P_t V_+ P_t),$$

we can write (2.9) as

$$\hat{u} = -\sum E(\omega_i)(I + P_{t_i} V_+ P_{t_i})^{-1}(I + P_{t_i} V_- P_{t_i})^{-1}\, G_{t_i} f_{t_i}.$$

Using the causality of V_+, it is easy to verify that $W_+ = (I + V_+)^{-1} - I$ satisfies the identity

$$P_t W_+ P_t = (I + P_t V_+ P_t)^{-1} - I$$

for all t. Hence,

$$\begin{aligned} \hat{u} = &-\sum E(\omega_i)(I + P_{t_i} V_- P_{t_i})^{-1}\, P_{t_i}(GT)^{\#}\, G_{t_i} f_{t_i} \\ &-\sum E(\omega_i)\, W_+ (I + P_{t_i} V_- P_{t_i})^{-1}\, P_{t_i}(GT)^{\#}\, G_{t_i} f_{t_i}. \end{aligned} \tag{2.11}$$

We claim that the norm of the second sum above is bounded by $M\varepsilon$ where M is a constant independent of the choice of partition. From continuity of the inverse and the fact that V_-, $(V_-)^* \in L^\mu$, it follows that $t \to (I + P_t V_- P_t)^{-1}$ is continuous. Thus $t \to x_t$ is continuous where $x_t = (I + P_t V_- P_t)^{-1} P_t (GT)^{\#}\, G_t f_t$. Since $[t_0, t_f]$ is compact, $\sup_t |x_t| = \alpha < \infty$. Thus for the second term we obtain the bound

$$\begin{aligned} |\sum E(\omega_i)\, W_+ x_{t_i}|^2 &= |\sum E(\omega_i)\, W_+ E(\omega_i)\, x_{t_i}|^2 \\ &\leqslant \sum |E(\omega_i)\, W_+ E(\omega_i)\, x_{t_i}|^2 \\ &\leqslant \alpha^2 \sup_{|x| = 1} \sum |E(\omega_i)\, W_+ E(\omega_i)\, x|^2 \\ &\leqslant \alpha^2 |W|_\mu^2 \max_i \mu(\omega_i). \end{aligned}$$

Thus we can choose $M = \alpha |W|_\mu$, which is independent of the partition. Noting that

$$\sum E(\omega_i)(I + P_{t_i} V_- P_{t_i})^{-1}\, P_{t_i}(GT)^{\#}\, G_{t_i} f_{t_i} = \sum E(\omega_i)(I + V_-)(GT)^{\#} G_{t_i} f_{t_i},$$

from (2.10) and (2.11) it follows that

$$|\hat{u} + \mathbf{F}_\mu((I + V_-)^{-1} A)\, z(\cdot)| \leqslant (1 + M)\, \varepsilon. \quad \blacksquare$$

The least squares argument used in [1] to derive the open loop control law for (2.1)–(2.2) also yields the following expression for the resulting cost.

THEOREM 2.8. *Let u, x denote the optimal pair for* (2.1)–(2.2). *Then*

$$J(u, x) = \tfrac{1}{2}\langle \{I - GT(I + (GT)^{\#} GT)^{-1}(GT)^{\#}\} f, f\rangle_X.$$

As can be discerned from the arguments in this section, the transition from open loop to quasifeedback was accomplished by use of a factorization argument (Theorem 1.3) and the principle of optimality (Theorem 2.6). As we shall see in the succeeding section, the transition from quasifeedback to a closed loop form is essentially a "substitution" using Proposition 1.1.

We also note that the quasifeedback form of the control law depends only on the "input-output" map of the system and does not require a "state-space" representation. This property partially extends to the feedback representation itself. This point will be further amplified when we discuss the hereditary control problem in the next section.

3. HEREDITARY SYSTEMS

In this section we shall consider control problems with dynamics

$$\begin{aligned} \dot{x}(t) &= \int_{-r}^{0} d_\theta\, \eta(t, \theta)\, x(t+\theta) + (BP_0 u)(t) \qquad && 0 \leqslant t \leqslant T \\ x(t) &= \phi(t) && t \in [-r, 0] \end{aligned} \tag{3.1}$$

where $x(\cdot)$, $\phi(\cdot) \in C([-r, 0], R^N) = X$, $u \in L_2([-r, T], R^M) = U$, $U' = L_2([-r, T], R^N)$, $B \in B(U, U')$.

The matrix valued function η is assumed measurable on $R \times R$ and is normalized so that $\eta(t, \theta) = 0$ for $\theta \geqslant 0$, $\eta(t, \theta) = \eta(t, -r)$ for $\theta \leqslant -r$. It is further assumed that $\eta(t, \cdot)$ is left continuous for each t, and that there exists a function $m \in L_1(-r, T)$ such that

$$|\mathrm{Var}\ \eta(t, \cdot)| \leqslant m(t)$$

where $|\cdot|$ denotes any matrix norm. The only restriction we place on the operator B is that it be causal. The cost functional associated with (3.1) is defined as

$$J(u, x) = \tfrac{1}{2}\int_{-r}^{T} \langle x(s), Q(s)\, x(s)\rangle\, d\mu(s) + \tfrac{1}{2}\int_{-r}^{T} \langle u(s), u(s)\rangle\, ds \tag{3.2}$$

where the inner products denote the Euclidean inner products, μ is an arbitrary positive regular Borel measure on $[-r, T]$ and $Q(s) \geqslant 0$, μ-a.e. s and is μ-essentially bounded.

The form of the problem above most often treated in the literature [7, 8, 9, 10, 11] consists of a finite number of constant delays in the state plus an integral term, no delays in the control and a cost defined as

$$\langle x(T), Qx(T)\rangle + \int_0^T [\langle x(t), Q(t)\, x(t)\rangle + \langle u(t), u(t)\rangle]\, dt.$$

This restricted class of problems can effectively be handled as a regulator problem in the space $M_2 = R^N \times L_2([-r, 0], R^N)$. However, the introduction of delays into the control complicates the situation considerably (see [12, 19, 20]).

We consider the more general problem formulation to demonstrate the generality of the methodology developed in Section 2. For example, the presence of arbitrary bounded control delays introduces no new difficulties into the derivation of the control law, i.e., the state arises in a straightforward manner from the quasifeedback representation. We will not however derive a Riccati equation here as the problem will remain formulated in $C([-r, T], R^N)$ and not in a space where (3.1) has an evolution representation. Rather, the feedback operator will be given in terms of the kernel of a Hilbert–Schmidt operator. This form is similar to one derived by Manitius [13] in a special case of (3.1)–(3.2). (We will have more to say about this and the relationship with the M_2 state-space formulation later.) The inclusion of the somewhat general cost has some practical value. For example, discrete state penalty terms which correspond to measures with nonzero singular part have applications in economics and engineering processes [14]. Also the flexibility in choice of measure may hold some promise in terms of approximating the feedback gain in applications.

In the sequel, we shall take the standard truncation resolution on U, and on X we define the strongly continuous family of projections P_X^t by

$$P^t x\colon s \to \begin{cases} x(s) & s < t \\ x(t) & s \geq t. \end{cases} \tag{3.3}$$

We note that the complementary projections P_t on X have the form

$$P_t x\colon s \to \begin{cases} 0 & s < t \\ x(s) - x(t) & s \geq t. \end{cases}$$

Integrating (3.1), it follows that

$$x(t) = f(t) + \int_0^t \int_{-r}^0 d_\theta \eta(s, \theta)\, x(\theta)\, ds + \int_0^t (BP_0 u)(s)\, ds,$$

where $f = P_X^0 \phi$. Thus the equation above is of the general form

$$x = f + Lx + Tu$$

with

$$Lx\colon t \to \begin{cases} 0 & t \in [-r, 0] \\ \int_0^t \int_{-r}^0 d_\theta \eta(s, \theta)\, x(\theta)\, ds & t \geq 0, \end{cases}$$

$$Tu\colon t \to \begin{cases} 0 & t \in [-r, 0] \\ \int_0^t (BP_0 u)(s)\, ds & t \geq 0. \end{cases}$$

Also it is evident that $P_X^\alpha L = 0$ for $\alpha \leq 0$. Furthermore, L is Volterra in the sense of Neustadt [15], thus satisfying Hypothesis 2.2.

The following variation of constants formula [16] will be needed:

$$x(t) = Y(t, 0)\, \phi(0) + \int_{-r}^0 d_\beta \left\{ \int_0^t Y(t, \alpha)\, \eta(\alpha, \beta - \alpha)\, d\alpha \right\} \phi(\beta)$$
$$+ \int_0^t Y(t, \sigma)(BP_0 u)(\sigma)\, d\sigma \tag{3.4}$$

where

$$Y(t, \sigma) = \begin{cases} 0 & t < \sigma \\ I - \int_\sigma^t Y(t, \alpha)\, \eta(\alpha, \sigma - \alpha)\, d\alpha & t \geq \sigma. \end{cases}$$

Furthermore, $Y(\cdot, \sigma)$ is absolutely continuous on $[\sigma, T]$ for each σ, $Y(t, \cdot)$ is of bounded variation for each t, and $\sup_{t,s} |Y(t, s)| < \infty$.

From (3.4) it follows that

$$GTu\colon t \to \int_{-r}^t Y(t, \sigma)(BP_0 u)(\sigma)\, d\sigma.$$

We now develop a kernal representation for this operator.

LEMMA 3.1. *There exists a measurable matrix valued function $F(t, s)$ on $[-r, T] \times [-r, T]$ such that*

$$GTu\colon t \to \int_{-r}^t F(t, s)\, u(s)\, ds \tag{3.5}$$

and

$$\sup_{t \in [-r, T]} \int_{-r}^{t} |F(t, s)|^2 \, ds < \infty. \tag{3.6}$$

Proof. First note that $\Phi \in B(U')$ defined by

$$\Phi u: t \to \int_{-r}^{t} Y(t, \sigma) \, u(\sigma) \, d\sigma \tag{3.7}$$

is Hilbert–Schmidt, since $\sup_{t,s} |Y(t, s)| < \infty$. Thus by the two-sided ideal property of Hilbert–Schmidt maps, $\Phi B P_0$ is necessarily an integral operator with kernel $F(t, s)$ that may be represented as (c.f., [17])

$$F(t, s) = [P_0 B^* Y'(t, \cdot)]'(s).$$

Thus as a mapping on $B(U, U')$,

$$GTu: t \to \int_{-r}^{t} F(t, s) \, u(s) \, ds. \tag{3.8}$$

Now since $Y(\cdot, \sigma)$ is absolutely continuous for each σ on $[\sigma, T]$, an application of the dominated convergence theorem shows that $t \to Y'(t, \cdot)$ is continuous as a function with values in $L_2([-r, T], R^{N \times N})$. Thus, $t \to F(t, \cdot)$ is also continuous as a function with values in $L_2([-r, T], R^{M \times N})$, and (3.6) is verified. Now again by the continuity of $t \to F(t, \cdot)$, it follows that the operator defined in (3.8) has range in X. Thus (3.5) also holds. ∎

LEMMA 3.2. *Let $H = L_2([-r, T], R^N; \mu)$. Then $(GT)^{\#} = A\pi$, where π is the continuous injection of X into H and $A \in B(H, U)$ is defined by*

$$Ax: t \to \int_{t}^{T} F'(s, t) \, Q(s) \, x(s) \, d\mu(s). \tag{3.9}$$

Furthermore, A is Hilbert–Schmidt.

Proof. Let $u \in U$, $x \in X$. Then from Lemma 3.1,

$$\begin{aligned} \langle GTu, x \rangle_X &= \int_{-r}^{T} \left\langle \int_{-r}^{t} F(t, \sigma) \, u(\sigma) \, d\sigma, Q(t) \, x(t) \right\rangle d\mu(t) \\ &= \int_{-r}^{T} \int_{-r}^{t} \langle u(\sigma), F'(t, \sigma) \, Q(t) \, x(t) \rangle \, d\sigma \, d\mu(t) \\ &= \int_{-r}^{T} \left\langle u(\sigma), \int_{\sigma}^{T} F'(t, \sigma) \, Q(t) \, x(t) \, d\mu(t) \right\rangle d\sigma. \end{aligned}$$

Thus we can define $(GT)^{\#} \in B(X, U)$ by

$$(GT)^{\#} x: t \to \int_{[t, T]} F'(s, t)\, Q(s)\, x(s)\, d\mu(s),$$

where for specificity we have written the integral over the closed interval $[t, T]$. (Note that $(t, T]$ would serve as well.) The operator A defined in (3.9) is then the extension of $(GT)^{\#}$ to H. Also note that the injection π of X into H is continuous since μ is finite. Hence $(GT)^{\#} = A\pi$. Furthermore, noting that

$$\int_r^T \int_{-r}^T |F'(s, t)\, Q(s)|^2\, d\mu(s)\, dt = \int_{-r}^T \int_{-r}^T |F'(s, t)\, Q(s)|^2\, dt\, d\mu(s)$$
$$\leqslant M|Q|_\infty\, \mu\{[-r, T]\}$$

where $M = \sup_t \int |F(t, s)|^2\, ds$ and $|Q|_\infty = \operatorname{ess\,sup} |Q(t)|$ (with respect to μ), it follows that A is Hilbert–Schmidt. (The proof of this last remark is the same as in the standard case where μ is also Lebesgue measure.) ∎

Lemmas 3.1 and 3.2 together imply that $(GT)^{\#} GT$ is Hilbert–Schmidt and that A is bounded by an absolutely continuous measure. Therefore Hypotheses (2.2) and (2.3) are satisfied in the present example. (All Hilbert–Schmidt maps can be bounded by an absolutely continous measure, cf. [2].)

We now apply Theorem 2.7 to obtain the feedback form for the optimization problem (3.1)–(3.2). In equations (3.10)–(3.12) below, $F(t, s)$ denotes the function defined in Lemma 3.1 and $W(t, s)$ is the kernel of the integral operator

$$W^* = (I + V^*)^{-1} - I$$

where V is obtained from the Volterra factorization

$$I + (GT)^{\#} GT = (I + V^*)(I + V).$$

THEOREM 3.3. *The optimal feedback control for* (3.1)–(3.2) *is given by*

$$\hat{u}(t) = -P(t, t)\, \hat{x}(t) - \int_t^{\min(t+r, T)} P(t, \alpha) \left\{ \int_{t-r}^{t} d_\beta \eta(\alpha, \beta - \alpha)\, \hat{x}(\beta) \right\} d\alpha$$
$$- \int_t^T P(t, \alpha)(BP_0 P' \hat{u})(\alpha)\, d\alpha, \qquad (3.10)$$

where

$$P(t, \alpha) = \int_{\alpha}^{T} K(t, s)\, Y(s, \alpha)\, d\mu(\alpha) \tag{3.11}$$

$$K(t, s) = \left[F'(s, t) + \int_{t}^{s} W(t, \sigma)\, F'(s, \sigma)\, d\sigma \right] Q(s). \tag{3.12}$$

Furthermore $P(t, \alpha)$ is measurable and square integrable on both the diagonal and the square $[0, T] \times [0, T]$; and also satisfies the "abstract" Wiener–Hopf (or Gelfand–Levitan) equation

$$P = [(GT)^{\#} \Phi]_{-} - \{[PBP_0]_{-} (GT)^{\#} \Phi\}_{-}, \tag{3.13}$$

where P is the Hilbert–Schmidt operator with kernel $P(t, \alpha)$ and Φ is defined as in (3.7).

Proof. From Theorem 2.7 we obtain the quasifeedback control

$$\hat{u} = -\mathbf{F}_{\nu}((I + V^{*})^{-1} A)\, z(\cdot),$$

where we have assumed that $V, A < \nu \ll \lambda$ (such a ν exists), A is defined by (3.9), and V and V^{*} are the causal and anticausal factors in the factorization

$$(I + (GT)^{\#} GT) = (I + V^{*})(I + V). \tag{3.14}$$

Let W denote the resolvent of V, i.e.,

$$W = (I + V)^{-1} - I. \tag{3.15}$$

Since $(GT)^{\#} GT$ is Hilbert–Schmidt, so is W^{*} (see [18]). Let $W(t, s)$ denote the kernel of W^{*}. Then $(I + V^{*})^{-1} A \in B(H, U)$ has the representation

$$(I + V^{*})^{-1} Ax \colon t \to \int_{t}^{T} K(t, \sigma)\, x(\sigma)\, d\sigma,$$

where

$$K(t, s) = \left[F'(s, t) + \int_{t}^{s} W(t, \sigma)\, F'(s, \sigma)\, d\sigma \right] Q(s).$$

Since $C([-r, T], H) \to L_2([-r, T], H; \nu)$, Proposition 1.1 implies

$$\hat{u}(t) = -\int_{t}^{T} K(t, s)\, z(t)(s)\, d\mu(s).$$

We now compute $z(t)(s)$ for $s \geqslant t$. Recalling the definition of P'_X from (3.3) and the definition of f_t from (2.8), we have

$$f_t(s) = \begin{cases} x(s) & s < t \\ \int_t^s (BP_0 P'\hat{u})(\sigma)\, d\sigma + x(t) & s \geqslant t. \end{cases}$$

From (3.4) it follows that for $s \geqslant t$,

$$(\pi G_t f_t)(s) = Y(s, t)\,\hat{x}(t) + \int_{t-r}^{t} d_\beta \left\{ \int_t^s Y(s, \alpha)\, \eta(\alpha, \beta - \alpha)\, d\alpha \right\} \hat{x}(\beta)$$
$$+ \int_t^s Y(s, \sigma)(BP_0 P'\hat{u})(\sigma)\, d\sigma.$$

Thus,

$$\hat{u}(t) = -\int_t^T K(t, s)\, Y(s, t)\, \hat{x}(t)\, d\mu(s)$$
$$-\int_t^T K(t, s) \int_t^s Y(s, \alpha) \left\{ \int_{t-r}^{t} d_\beta \eta(\alpha, \beta - \alpha)\, \hat{x}(\beta) \right\} d\alpha\, d\mu(s)$$
$$-\int_t^T K(t, s) \int_s^t Y(s, \sigma)(BP_0 P'\hat{u})(\sigma)\, d\sigma\, d\mu(s),$$

where we have interchanged the order of integration in the second integral using the unsymmetric Fubini theorem of Cameron and Martin (c.f., [16]). Now define

$$P(t, \alpha) = \int_\alpha^T K(t, s)\, Y(s, \alpha)\, d\mu(s) \qquad \alpha \geqslant t. \tag{3.16}$$

Substituting this into the control law above and using Fubini's theorem, it follows

$$\hat{u}(t) = -P(t, t)\,\hat{x}(t) - \int_t^{\min(t+r, T)} P(t, \alpha) \int_{t-r}^{t} d_\beta \eta(\alpha, \beta - \alpha)\, \hat{x}(\beta)\, d\alpha$$
$$-\int_t^T P(t, \alpha)(BP_0 P'\hat{u})(\alpha)\, d\alpha,$$

and (3.10) is verified. The measurability of $P(t, t)$ and $P(t, \alpha)$ follow from (3.16) and Fubini's theorem. Now since $\sup_{\alpha, s} |Y(s, \alpha)| < \infty$,

$$|P(t, \alpha)| \leqslant M \int_{-r}^{T} |K(t, s)|\, d\mu(s)$$

for some constant M. Also,

$$|K(t, s)| \leqslant \left\{|F'(s, t)| + \left|\int_t^s W(t, \sigma) F'(x, \sigma)\, d\sigma\right|\right\} |Q(s)|$$
$$\leqslant \{|F'(s, t)| + M_2 g(t)\}|Q(s)|$$

where $M_2 = \sup_s [\int |F'(s, \sigma)|^2\, d\sigma]^{1/2}$ (cf., Lemma 3.1) and

$$g(t) = \left[\int |W(t, \sigma)|^2\, d\sigma\right]^{1/2}.$$

Noting that $g \in L_2(-r, T)$ since W is Hilbert–Schmidt, we have

$$\begin{aligned}\int |P(t, t)|^2\, dt &\leqslant M^2 \int \left[\int |K(t, s)|\, d\mu(s)\right]^2 dt \\ &\leqslant M^2(T+r)^2 \iint |K(t, s)|^2\, d\mu(s)\, dt \\ &\leqslant M^2(T+r)^2 \iint \{|F'(s, t)|^2 + M_2 |F'(s, t)|\, g(t) + M_2^2 g^2(t)\} \\ &\quad \times |Q(s)|^2\, d\mu(s)\, dt \\ &< \infty,\end{aligned}$$

where we have used Lemma 3.1 and Fubini's theorem. Essentially the same argument shows that

$$\iint |P(t, \alpha)|^2\, d\alpha dt < \infty.$$

To prove (3.11), we introduce the Hilbert–Schmidt operator $\tilde{P}$ as

$$\tilde{P} = (I + V^*)^{-1} (GT)^{\#} \Phi. \tag{3.17}$$

Noting that $\tilde{P}$ is a Fredholm integral operator with kernel

$$\tilde{P}(t, \alpha) = \int_\alpha^T K(t, s)\, Y(s, \alpha)\, d\mu(s),$$

(3.16) implies that $\tilde{P}(t, \alpha) = P(t, \alpha)$ for $\alpha \geqslant t$. Now, the factorization (3.14) implies

$$\begin{aligned}(GT)^{\#} GT &= V + V^* + V^* V \\ &= V^* + (I + V^*)\, V,\end{aligned}$$

so that

$$V = (I + V^*)^{-1}(GT)^{\#}GT - V^*.$$

Taking the causal parts of both sides in the above, we obtain

$$V = [(I + V^*)^{-1}(GT)^{\#}GT]_+.$$

Noting that $\Phi BP_0 = GT$, (3.17) and (3.18) then imply

$$[\tilde{P}BP_0]_- + V = (I + V^*)^{-1}(GT)^{\#}GT.$$

Thus,

$$(I + V)^{-1}\{[\tilde{P}TP_0]_- + V\} = (I + V)^{-1}(I + V^*)^{-1}(GT)^{\#}GT.$$

But since $(I + V)^{-1}(I + V^*)^{-1} = (I + (GT)^{\#}GT)^{-1}$, the expression above can be written as

$$[\tilde{P}BP_0]_- = -(I + V)\, R - V,$$

where $R = (I + (GT)^{\#}GT)^{-1} - I$. Then by taking anticausal parts we arrive at

$$[\tilde{P}BP_0]_- = -[(I + V)\, R] \quad . \tag{3.19}$$

Now inverting both sides of (3.14) and substracting the identity results in

$$R = W + W^* + WW^*,$$

where W is defined in (3.15). Hence,

$$W^* = (I + V)\, R - W.$$

Then by taking anticausal parts in the above, we have

$$W^* = [(I + V)\, R]_-.$$

Substituting this into (3.19), we obtain

$$[\tilde{P}BP_0]_- = -W^*.$$

Since $(I + V^*)^{-1} = I + W^*$, (3.17) now becomes

$$\tilde{P} = (GT)^{\#}\Phi - [\tilde{P}BP_0]_-(GT)^{\#}.$$

Defining $P = \tilde{P}_-$ and recalling that B is causal, it follows

$$P = [(GT)^{\#}\Phi]_- - \{[PBP_0]_-(GT)^{\#}\Phi\}. \quad \blacksquare$$

The derivation of the feedback control law from the quasifeedback form naturally results in the correct state, i.e., the inclusion of a portion of the histories of both the trajectory and the control. For example, if the causal operator B has the form:

$$(Bu)(t) = \sum_{i=1}^{n} B_i(t)\, u(t-d_i) + \int_{-r}^{t} k(t-s)\, u(s)\, ds$$

with $0 \leqslant d_1 \leqslant \cdots \leqslant d_n \leqslant r$, k integrable, $k(t)=0$ for $t>\delta \geqslant 0$, then the dependence on the past control in (3.10) can be written as

$$\begin{aligned}\int_t^T P(t,\alpha)(BP_0P'u)(\alpha)\, d\alpha = {} & \sum_{i=1}^{n} \int_{\max(t-d_i,0)}^{\min(t,T-d_i)} P(t,\alpha+d_i)\, B(\alpha+d_i)\, \hat{u}(\alpha)\, d\alpha \\ & + \int_{\max(t-\delta,0)}^{t} \left\{ \int_{\alpha}^{\min(\alpha+\delta,T)} P(t,\sigma)\, k(\sigma-\alpha)\, d\sigma \right\} \\ & \times \hat{u}(\alpha)\, d\alpha.\end{aligned}$$

When $d_n=\delta=0$, the expression above is zero and the state does not include the past control.

We note that (3.10) can in general be solved directly for u in terms of x, to obtain a causal controller only dependent on x. To see this write the equation as

$$u(t) = \xi(t) - \int_t^T P(t,\alpha)(BP_0P'u)(\alpha)\, d\alpha, \tag{3.20}$$

where

$$\xi(t) = -P(t,t)\, x(t) - \int_t^{\min(t+r,T)} P(t,\alpha) \int_{t-r}^{t} d_\beta \eta(\alpha, \beta-\alpha)\, x(\beta)\, d\alpha.$$

Recognizing the operator in (3.20) defined by the integral as $[PB]_+$, we obtain

$$(I + [PBP_0]_+)\, u = \xi.$$

But since P is Hilbert–Schmidt, and $[PBP_0]_+$ is Hilbert–Schmidt and causal, it necessarily follows that $[PBP_0]_+$ is quasinilpotent. Thus,

$$u = (I + [PBP_0]_+)^{-1}\, \xi.$$

Recall that an element $\tilde{v} \in M_2$ has the form $\tilde{v} = (v_0, v(\cdot))$ where $v_0 \in R^n$ and $v(\cdot) \in L_2\ ((-r, 0), R^n)$. In the M_2 formulation of the control problem (in the time invariant case without control delays, and Lebesgue measure in (3.2)), the solution to (3.1) is represented as

$$\tilde{x}(t) = S(t)\,\tilde{x}(0) + \int_0^t S(t-\sigma)\,\tilde{B}u(\sigma)\,d\sigma$$

and the cost is expressed

$$J(u, \tilde{x}) = \int_{t_0}^{t_f} \{\langle \tilde{x}(t), \tilde{Q}\,\tilde{x}(t)\rangle + |u(t)|^2\}\,dt.$$

Here $S(\cdot)$ is the solution semigroup on M_2, and $\tilde{B} \in B(R^n, M_2)$ and $\tilde{Q} \in B(M_2)$ are defined by

$$\tilde{B}u = (Bu, 0),\ \tilde{Q}(v_0, v(\cdot)) = (Qv_0, 0).$$

The input-output map for this system has the form

$$Tu\colon t \to \tilde{Q}^{1/2} \int_0^t S(t-\sigma)\,\tilde{B}u(\sigma)\,d\sigma.$$

If we let P denote te projection in $B(M_2)$ such that $P(v_0, v(\cdot)) = (v_0, 0)$, then the input-output map can be equivalently represented as

$$Tu\colon t \to \tilde{Q}^{1/2} \int_0^t P\,S(t-\sigma)\,P\tilde{B}u(\sigma)\,d\sigma.$$

Thus only the restriction of the semigroup to the canonical finite dimensional "piece" of M_2 enters the input-output formulation. Then specifically the property we have exploited is that $PS(t)\,P(v_0, v(\cdot)) = (Y(t, 0)\,v_0, 0)$. Hence our representation of the optimal feedback gain does not require the entire semigroup, but only that part that contributes to the input-output map.

This point has some ramifications in terms of numerical approximations of the gain $P(t, \alpha)$. In the M_2 state space formulation, a delicate analysis of the infinitesimal generator of $S(\cdot)$ and its adjoint are required to obtain strong convergence of the Riccati operators. This in turn results in essentially uniform convergence of approximations to $P(t, t)$ and L_2-convergence of approximations to $P(t, \alpha)$ [10]. The situation becomes more complex when the system is time-varying [11]. And the introduction of delays into the control produces an infinite dimensional input space with unbounded "B" map, thus rendering the approximation results in [10], [11] not directly applicable. (In fact as far as we know, time varying systems with general time varying control delays have not been treated in the literature.) Although we have not pursued an algorithmic form for the gain, and it remains to be seen what properties such an algorithm might possess, the quantities involved in the definition of $P(t, \alpha)$ are in principle computable. The matrix $Y(t, s)$ is the solution of a linear Volterra equation, and $F(t, s)$ is then readily available once B is specified. The only nonlinear calculation involves the determination of $W(t, s)$ from the factorization of $I+(GT)^{\#}GT$. A numerical method for the factorization involving continuous kernels has been previously developed [27]. Thus approximations of $P(t, \alpha)$ are at least amenable to analysis.

We finally note that the Gelfand–Levitan equation (3.13), in the case of no control delays, becomes

$$P(t, s) = H(t, s) - \int_t^T P(t, \sigma)\, B(\sigma)\, H(\sigma, s)\, d\sigma \tag{3.21}$$

where

$$H(t, s) = \int_{\max(t,s)}^T F'(\sigma, t)\, Q(\sigma)\, Y(\sigma, s)\, d\mu(\sigma).$$

The equation above was derived by Manitius in the case of simple state delays (and where μ was Lebesgue measure plus a point mass at T). His derivation required sufficient continuity hypotheses to ensure the validity of certain point evaluations. The use of the factorization circumvents these difficulties in the general situation. Also we hasten to add that (3.21) is not a Volterra equation, but a parametrized family of Fredholm equations. As is well known, the Gelfand–Levitan equation (3.21) is intimately connected with the problem of factorization, and in fact its solution is given by (3.16), which involves the factor V.

References

1. M. Milman and A. Schumitzky, A vector measure approach to state feedback control in Banach resolution spaces. I., *J. Math. Anal. Appl.* **90** (1982), 83–103.
2. M. Milman and A. Schumitzky, On a class of operators on Hilbert space with application to factorization and systems theory, *J. Math. Anal. Appl.* (1984).
3. J. Foster, "Optimal Control of Hereditary Systems," Ph.D. thesis, Univ. of Southern California, Los Angeles, 1976.
4. M. Milman, "Feedback Control of Hereditary Systems Governed by Linear Operator Equations," Ph.D. thesis, Univ. of Southern California, Los Angeles, 1979.
5. R. M. DeSantis, R. Saekes, and L. Tung, Basic optimal estimation and control problems in Hilbert space, *Math. Systems Theory* **12** (1978), 175–203.
6. A. Schumitzky, State feedback control for general linear systems, *in* "Proceedings of the Inter. Sympos. on the Math. of Networks and Systems," Vol. 3 (1979), pp. 194–200.

7. M. C. Delfour and S. K. Mitter, Controllability, observability and optimal feedback control for affine hereditary systems, *SIAM J. Control Optim.* **10** (1972), 298–327.
8. M. C. Delfour, State theory of linear hereditary differential systems, *J. Math. Anal. Appl.* **60** (1977), 8–35.
9. H. T. Banks and T. A. Burns, Hereditary control problems: Numerical methods based on averaging approximations, *SIAM J. Control Optim.* **16** (1978), 169–208.
10. J. S. Gibson, Linear-quadratic optimal control of hereditary differential systems: infinite dimensional Riccati equations and numerical approximations, *SIAM J. Control Optim.* **21** (1983), 95–139.
11. K. Kunisch, Approximation schemes for the linear-quadratic optimal control problem associated with delay equations, *SIAM J. Control Optim.* **20** (1982), 506–540.
12. A. Ichikawa, Quadratic control of evolution equations with delays in control, *SIAM J. Control Optim.* **20** (1982), 645–668.
13. A. Manitius, Optimal control of linear time-lag processes with quadratic performance indexes *in* "Proc. of 4th IFAC Congress," Warsaw, Poland, 1969, pp. 16–28.
14. H. P. Geering, Continuous-time optimal control theory for cost functionals including discrete state penalty terms, *IEEE Trans. Automat. Control* **AC-21** (1976).
15. L. Neustadt, On the solutions of certain integral-like operator equations. Existence, uniqueness, and dependence theorems, *Archive Rational Mech. Anal.* **38** (1970), 131–160.
16. J. Hale, "Theory of Functional Differential Equations," Springer-Verlag, New York/Berlin, 1977.
17. A. C. Zaanen, "Linear Analysis," Interscience, New York, 1953.
18. I. C. Gohberg and M. G. Krein, "Theory and Applications of Volterra Operators in Hilbert Space," Amer. Math. Soc., Providence, R. I., 1970.
19. M. C. Delfour, "The Linear Quadratic Optimal Control Problem with Delays in the State and Control Variables: A State Space Approach," Centre de Recherche de Mathematiques Appliquees, University of Montreal, CRM-1012, Canada, 1985.
20. K. Ito, "Regulator Problem for Hereditary Differential Systems with Control Delays," ICASE Report 82-3, NASA Langley Research Center, Hampton, Va, 1982.
21. R. B. Vinter and R. H. Kwong, The infinite quadratic control for linear systems with state and control delays: An evolution equation approach, *SIAM J. Control Optim.* **19** (1981), 139–153.
22. R. Bellman, "Dynamic Programming," Princeton Univ. Press, Princeton, N. J., 1957.
23. R. Bellman, Functional equations in the theory of dynamic programming. VII: A partial differential equation for the Fredholm resolvent, *Proc. Amer. Math. Soc.* **8** (1957), 435–440.
24. I. M. Gelfand and B. M. Levitan, On the determination of a differential equation from its spectral function, "Amer. Math. Soc. Translations, Ser. 2," 1955, pp. 253–304.
25. M. G. Krein, A new method of solution of linear integral equations of the first and second kinds, *Dokl. Akad. Nauk. SSSR* **100** (1955), 413–416.
26. A. J. F. Siegert, A systematic approach to a class of problems in the theory of noise and other random phenomena. II: Examples, *IRE Trans. Inform. Theory*, **IT-3** (1957), pp. 38–43.
27. A. McNabb and A. Schumitzky, Factorization of operators. II: A nonlinear Volterra method for numerical solution of linear Fredholm equations, *J. Comput. System Sci.* **4** (1970), 103–128.